QUANTUM PHYSICS I - Nov. 4, 2021

Write your name and student number on **all** sheets. There are 4 problems in this exam. All subquestions are separately 5 points, with 90 points in total. You are allowed to use the Griffiths book for consultation.

Problem 1: HARMONIC OSCILLATOR (25 points)

Consider the harmonic oscillator in three dimensions, with potential energy

$$V = \tfrac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)\,,$$

in terms of Cartesian frequencies ω_x, ω_y and ω_z.

a) What is the commutator between the ladder operators a_x^+ and a_y^- (the raising and lowering operators with respect to x and y, respectively)?

b) Suppose all the frequencies are equal, $\omega_x = \omega_y = \omega_x$. What is the general formula for the energy spectrum in terms of the Cartesian quantum numbers n_x, n_y, n_z?

c) What is the degeneracy of the second excited state of the total system? Explain how this arises in terms of the Cartesian quantum numbers.

d) Write down an example of the wavefunction of a second excited state, including the explicit space- and time-dependence (but not normalisation).

e) What happens to the energy of the second excited state of question c) when the frequencies of one of the directions, say ω_x, suddenly increases by 10%? Sketch the resulting energy levels and their degeneracies.

Problem 2: ANGULAR MOMENTUM (20 points)

The operator corresponding to angular momentum in three dimensions has three components. It is customary to form ladder operators from linear combinations of two of these components,

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\frac{\cos(\theta)}{\sin(\theta)}\frac{\partial}{\partial\phi}\right)\,.$$

a) Calculate the operator L_- on the test function $\sin^2(\theta)e^{-2i\phi}$. Interpret your results in terms of ladder operators and spherical harmonics.

b) Calculate the operator L_+ on the above test function. Interpret your results in terms of ladder operators and spherical harmonics.

c) Using the previous two results, calculate the combination $L_+L_- - L_-L_+$ on the same test function. What should the commutator of the ladder operators be and how does this compare to your result?

d) Is the spherical harmonic Y_1^1 an eigenstate of L_x and L_y? Briefly explain your answer using the Heisenberg uncertainty principle for angular momenta.

Problem 3: DELTA-FUNCTION POTENTIALS (25 points)

Consider the following potential for a one-dimensional quantum-mechanical system:

$$V = -\alpha\delta(x) - \beta\delta(x - x_0)\,.$$

a) Consider scattering states that have a wavenumber (and hence momentum) given by $k = n\pi/x_0$ with n an arbitrary integer. Write down the most general spatial wavefunction corresponding to an incoming scattering state from the left. Do not worry about normalisation, time-dependence and boundary conditions for the moment.

b) What are the boundary conditions to be imposed? Derive the transmission and reflection coefficients as a function of the two coefficients α and β.

c) For specific values of α and β, the total system can become transparent for these scattering states, that is the reflection coefficient vanishes. (If you did not find an answer at the previous questions, you can take for instance $R = \sin^2(\alpha + \beta)$ and $T = \cos^2(\alpha + \beta)$.) Explain in a few sentences how this is physically possible, and indicate which wave contributions are relevant for this result.

d) If you would probe the same system with different scattering states, that is wavenumbers that are different from $k = n\pi/x_0$, do you expect to have perfect transmission as well? Briefly explain your answer.

e) The system also has bound states. Sketch the form of the spatial profile of the ground state for two cases: $\alpha = \beta$ and $\alpha = 10\beta$ (all positive), and indicate the difference(s).

Problem 4: ENTANGLEMENT (20 points)

Consider a single particle that can be in either of two boxes (that can be on separate sides of the Universe). Denote the wavefunction of each box separately by $|1>$ when it contains the particle, and by $|0>$ when it does not.

a) Write down the most general (normalised) wavefunction that the total system (comprising two boxes and one particle) can be in.

b) Can the states of the two boxes be entangled? If so, give an example of a wavefunction (of the total system) that corresponds to an entangled state; if not, briefly explain your answer.

c) Can the states of the two boxes be non-entangled? If so, give an example of a wavefunction (of the total system) that corresponds to a non-entangled state; if not, briefly explain your answer.

d) One can generalize the situation from two boxes to an infinite amount of boxes, labelled by x. The total state of the system can then be described by a spatial wavefunction $\psi(x)$. Suppose this is given by for instance a Gaussian profile. Does the collapse of the wavefunction during a measurement of the location of the particle corresponds to 'spooky action at a distance', as Einstein referred to the implications of entanglement? Briefly explain your answer.